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Derivatives of multidimensional Bernstein operators and smoothness☆

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Abstract

We characterize the directional derivatives of multidimensional Bernstein operators by a new measure of smoothness. This task is carried out by means of establishing the relation between the asymptotic behavior of the derivatives and the smoothness of the functions they approximate. The obtained results generalize the corresponding ones for univariate Bernstein operators.

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1. Introduction

Let $S := S_d$ be the simplex in $\mathbb{R}^d (d \in \mathbb{N})$ defined by

$$S := \left\{ \mathbf{x} := (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i \geqslant 0,\, |\mathbf{x}| := \sum_{i=1}^{d} x_i \leqslant 1 \right\}.$$

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The Bernstein operators on S are given by

$$B_{n,d}f := B_{n,d}(f(\cdot),\mathbf{x}) := \sum_{|\mathbf{k}|\leqslant n} P_{n,\mathbf{k}}(\mathbf{x}) f\left(\frac{\mathbf{k}}{n}\right), \quad \mathbf{x}\in S, \quad n\in\mathbb{N}, \tag{1.1}$$

where $\mathbf{k} := (k_1, k_2, \ldots, k_d)$ with k_i non-negative integers, $|\mathbf{k}| := \sum_{i=1}^{d} k_i$, and

$$P_{n,\mathbf{k}}(\mathbf{x}) := \frac{n!}{\mathbf{k}!(n-|\mathbf{k}|)!}\mathbf{x}^{\mathbf{k}}(1-|\mathbf{x}|)^{n-|\mathbf{k}|}$$

with the convention

$$\mathbf{x}^{\mathbf{k}} := x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}, \quad \text{and} \quad \mathbf{k}! := k_1!k_2!\cdots k_d!.$$

For $d = 1$, the multivariate Bernstein operators given in (1.1) reduce to the classical Bernstein operators:

$$B_n(f,x) := B_{n,1}(f,x) := \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n}\right), \quad x\in[0,1]. \tag{1.2}$$

For the simplex S, we denote by V_S the set of unit vectors in the directions of the edges of S where $-\mathbf{e}$ and $\mathbf{e}$ are considered to be the same vector. We define the weight function, for direction $\zeta \in V_S$ and a point $\mathbf{x} \in S$, as (see also [6] and [11])

$$\varphi_\zeta^2(\mathbf{x}) := \inf_{\mathbf{x}+\lambda\zeta\notin S,\lambda>0} d(\mathbf{x},\mathbf{x}+\lambda\zeta) \inf_{\mathbf{x}-\lambda\zeta\notin S,\lambda>0} d(\mathbf{x},\mathbf{x}-\lambda\zeta),$$

where $d(\mathbf{x},\mathbf{y})$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^d$. Clearly,

$$\varphi_\zeta^2(\mathbf{x}) = \begin{cases} x_i(1-|\mathbf{x}|), & \zeta = \mathbf{e}_i, \quad 1\leqslant i\leqslant d;\\ 2x_ix_j, & \zeta = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}, \quad 1\leqslant i < j\leqslant d,\end{cases}$$

where $\mathbf{e}_i$ is the unit vector in $\mathbb{R}^d$, i.e., its ith component is 1 and the others are 0.

For $0\leqslant\lambda\leqslant 1$, $\zeta\in V_S$ and $r\in\mathbb{N}$, we define the K-functional as

$$K_r^\zeta(f,t)_{\varphi_\zeta^\lambda} := \inf_{g\in D_r(S)}\left\{\|f-g\|_{C(S_d)} + t^r\left\|\varphi_\zeta^{\lambda r}\left(\frac{\partial}{\partial\zeta}\right)^r g\right\|_{C(S_d)}\right\}, \quad t>0,$$

where

$$D_r(S) := \left\{g\in C(S) : g\in C^r(S_0), \text{ and } \varphi_\zeta^{\lambda r}\left(\frac{\partial}{\partial\zeta}\right)^r g\in C(S)\right\},$$

and S_0 is the interior of S.

To characterize the derivatives of multivariate Bernstein operators, we introduce a measure of smoothness defined by

$$\omega_r^\zeta(f,t)_{\varphi_\zeta^\lambda} := \sup_{0<h\leqslant t}\left\|\Delta_{h\zeta\varphi_\zeta^\lambda}^r f\right\|_{C(S_d)}, \quad f\in C(S), \quad \zeta\in V_S, \quad 0\leqslant\lambda\leqslant 1,$$

where

$$\Delta_{h\mathbf{e}}^r f(\mathbf{x}) := \begin{cases} \sum_{i=0}^r (-1)^i \binom{r}{i} f\left(\mathbf{x} + (r-i)h\mathbf{e}\right), & \mathbf{x}, \mathbf{x} + rh\mathbf{e} \in S; \\ 0, & \text{otherwise.} \end{cases}$$

In [3] Ditzian studied the relation between the derivatives of classical Bernstein operators $B_{n,1}f$ and the smoothness of function *f*; he proved that

Theorem 1.1. *Let* $r = 1, 2$, $\varphi^2(x) := x(1-x)$, *and assume that* $f \in C[0,1]$ *satisfies* $\omega_r(f,t) \leqslant Ct^\beta$ *for some* $\beta > 0$. *Then the following equivalence holds true for* $0 < \alpha < r$:

$$\left|B_{n,1}^{(r)}(f,x)\right| \leqslant C\left\{\min\left(n^2, \frac{n}{\varphi^2(x)}\right)\right\}^{(r-\alpha)/2}$$

if and only if

$$\omega_r(f,t) = \mathcal{O}(t^\alpha).$$

Here and in the sequel, $\omega_r(f,t)$ *is the classical modulus of continuity of r order for the univariate function f, C denotes the positive constant which is independent of n, f and* $\mathbf{x}$, *but its value may be different at different occurrence.*

Zhou [12] extended the result to the cases of higher order derivatives; he proved that Theorem 1.1 is valid for any $r \in \mathbb{N}$. We notice that in [7] the global result for higher order derivatives of univariate Bernstein–Durrmeyer operators was characterized in $L^p[0,1](1 \leqslant p \leqslant \infty)$ by Ditzian–Totik's modulus. Moreover, we know that Ditzian [5] used $\omega_2^{\mathbf{e}_1}(f,t)_{\varphi_{\mathbf{e}_1}^\lambda}$ (the case of $d = 1$ and $r = 2$ for $\omega_r^\zeta(f,t)_{\varphi_\zeta^\lambda}$) and gave an interesting direct estimate for univariate Bernstein operators. The estimate combines the classical local result (the case $\lambda = 0$) with the global norm result (the case $\lambda = 1$) developed by Ditzian and Totik [10]. Such results for univariate polynomial approximation were previously investigated in [8,9].

In this paper, we study the characterization of derivatives for multidimensional Bernstein operators by using the measure of smoothness $\omega_r^\zeta(f,t)_{\varphi_\zeta^\lambda}$. The main results are as follows:

Theorem 1.2. *Suppose* $r \in \mathbb{N}$, $f \in C(S)$, $0 < \alpha < r$, $0 \leqslant \lambda \leqslant 1$, $\zeta \in V_S$, *and* $\omega_r^\zeta(f,t)_{\varphi_\zeta^\lambda} = \mathcal{O}\left(t^\alpha\right)$. *Then we have*

$$\left|\varphi_\zeta^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial\zeta}\right)^r B_{n,d}(f,\mathbf{x})\right| \leqslant C\left\{\min\left(n^{2-\lambda}, \frac{n}{\varphi_\zeta^{2(1-\lambda)}(\mathbf{x})}\right)\right\}^{(r-\alpha)/2}.$$

Theorem 1.3. *Let* $r \in \mathbb{N}$, $\zeta \in V_S$, *and assume that* $f \in C(S)$ *satisfies* $\omega_r^\zeta(f,t) = \mathcal{O}\left(t^\beta\right)$ *for some* $\beta > 0$. *Then the following equivalence holds true for* $0 < \alpha < r$:

$$\left|\left(\frac{\partial}{\partial\zeta}\right)^r B_{n,d}(f,\mathbf{x})\right| \leqslant C\left\{\min\left(n^2, \frac{n}{\varphi_\zeta^2(\mathbf{x})}\right)\right\}^{(r-\alpha)/2},$$

if and only if

$$\omega_r^{\zeta}(f,t)=\mathcal{O}(t^{\alpha}).$$

2. Some lemmas

To prove Theorems 1.2 and 1.3 we will show some lemmas in this part.

Lemma 2.1. *Let* $f\in C(S)$, $r\in\mathbb{N}$, $0\leqslant\lambda\leqslant 1$ *and* $\zeta\in V_S$. *We have*

$$\left|\varphi_{\zeta}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial\zeta}\right)^{r}B_{n,d}(f,\mathbf{x})\right|\leqslant Cn^{r/2}\left(\max\left(n^{-1/2},\varphi_{\zeta}(\mathbf{x})\right)\right)^{r(\lambda-1)}\|f\|_{C(S_d)}.$$

Proof. First, we recall the discussion of Theorem 4.1 of [11] that will allow us to consider Lemma 2.1 for $\zeta=\mathbf{e}_1$ only in which case $\partial/\partial\zeta=\partial/\partial x_1$. It is clear that if $\zeta=\mathbf{e}_i$, $2\leqslant i\leqslant d$, we may just rename the coordinates. The following transformation will help us to complete the other cases of ζ. The transformation T (see p. 102–103 of [11]) defined by

$$T(x_1,x_2,\ldots,x_d):=(u_1,u_2,\ldots,u_d):=\mathbf{u},\quad u_l=x_l\quad\text{for}\quad l\neq j,\, u_j=1-|\mathbf{x}|$$

satisfies

$$T^2=I,\quad I\text{ is the identity operator, } T:S\to S\quad\text{onto},$$
$$\frac{\partial}{\partial u_l}=\frac{\partial}{\partial x_l}-\frac{\partial}{\partial x_j}\quad\text{for}\quad l\neq j,\ \frac{\partial}{\partial u_j}=-\frac{\partial}{\partial x_j}.$$

We then can check (see also [4])

$$B_{n,d}(f,\mathbf{x})=B_{n,d}(f_T,T\mathbf{x}),\quad B_{n,d}(f,T\mathbf{x})=B_{n,d}(f_T,\mathbf{x}),$$

where $f_T(\mathbf{u})=f(\mathbf{x})$, $\mathbf{u}=T\mathbf{x}$. So, for $\zeta=(\mathbf{e}_i-\mathbf{e}_j)/\sqrt{2}$, $1\leqslant i<j\leqslant d$, we have (see also p. 103 of [11])

$$\begin{aligned}\left|\varphi_{\zeta}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial\zeta}\right)^{r}B_{n,d}(f,\mathbf{x})\right|&=\left|\varphi_{\mathbf{e}_i}^{r\lambda}(\mathbf{u})\left(\frac{\partial}{\partial u_i}\right)^{r}B_{n,d}(f_T,\mathbf{u})\right|\\&\leqslant Cn^{r/2}\left(\max\left(n^{-1/2},\varphi_{\mathbf{e}_i}(\mathbf{u})\right)\right)^{r(\lambda-1)}\|f_T\|_{C(S_d)}\\&=Cn^{r/2}\left(\max\left(n^{-1/2},\varphi_{\zeta}(\mathbf{x})\right)\right)^{r(\lambda-1)}\|f\|_{C(S_d)}.\end{aligned}$$

Secondly, we prove

$$\left|\varphi_{\mathbf{e}_1}^{r}(\mathbf{x})\left(\frac{\partial}{\partial x_1}\right)^{r}B_{n,d}(f,\mathbf{x})\right|\leqslant Cn^{r/2}\|f\|_{C(S_d)}.\tag{2.1}$$

Let

$$\mathbf{x}^*:=(x_2,x_3,\ldots,x_d),\,\mathbf{x}=(x_1,\mathbf{x}^*)\in S,\quad \mathbf{k}^*:=(k_2,k_3,\ldots,k_d),\,\mathbf{k}=(k_1,\mathbf{k}^*),$$

we write

$$|\mathbf{x}^*| := \sum_{i=2}^{d} x_i, \quad |\mathbf{k}^*| := \sum_{i=2}^{d} k_i.$$

Recalling that $|\mathbf{x}^*| = 1$ implies $x_1 = 0$, we can put $y := \frac{x_1}{1-|\mathbf{x}^*|}$ for $0 \leqslant |\mathbf{x}^*| < 1$, and $y := 0$ for $|\mathbf{x}^*| = 1$. Therefore, $\varphi_{\mathbf{e}_1}(\mathbf{x})$ can be rewritten as

$$\varphi_{\mathbf{e}_1}(\mathbf{x}) = (1 - |\mathbf{x}^*|)\varphi(y)$$

and $P_{n,\mathbf{k}}(\mathbf{x})$ can be decomposed as

$$P_{n,\mathbf{k}}(\mathbf{x}) = P_{n,\mathbf{k}^*}(\mathbf{x}^*)P_{n-|\mathbf{k}^*|,k_1}(y).$$

Using the fact that

$$\left\| \sum_{k=0}^{n} \varphi^r(\cdot)P_{n,k}^{(r)}(\cdot) \right\|_{C[0,1]} \leqslant Cn^{r/2}$$

proved in [2], we know that

$$\begin{aligned}
\left| \varphi_{\mathbf{e}_1}^r(\mathbf{x}) \left(\frac{\partial}{\partial x_1} \right)^r B_{n,d}(f,\mathbf{x}) \right| &\leqslant \left| \sum_{|\mathbf{k}| \leqslant n} P_{n,\mathbf{k}^*}(\mathbf{x}^*)\varphi^r(y)P_{n-|\mathbf{k}^*|,k_1}^{(r)}(y) f\left(\frac{\mathbf{k}}{n}\right) \right| \\
&\leqslant C\|f\|_{C(S_d)} \max_{\mathbf{x}^* \in S_{d-1}} \sum_{|\mathbf{k}^*| \leqslant n} P_{n,\mathbf{k}^*}(\mathbf{x}^*) \\
&\quad \times \sup_{|\mathbf{k}^*| \leqslant n} \left\| \varphi^r(\cdot) \sum_{k_1=0}^{n-|\mathbf{k}^*|} P_{n-|\mathbf{k}^*|,k_1}^{(r)}(\cdot) \right\|_{C[0,1]} \\
&\leqslant Cn^{r/2}\|f\|_{C(S_d)}.
\end{aligned}$$

Also, it is easy to obtain

$$\begin{aligned}
\left| \left(\frac{\partial}{\partial x_1} \right)^r B_{n,d}(f,\mathbf{x}) \right| &= \left| \frac{n!}{(n-r)!} \sum_{|\mathbf{k}| \leqslant n-r} P_{n-r,\mathbf{k}}(\mathbf{x})\Delta_{\frac{1}{n}\mathbf{e}_1}^r f\left(\frac{\mathbf{k}}{n}\right) \right| \\
&\leqslant Cn^r\|f\|_{C(S_d)}.
\end{aligned} \tag{2.2}$$

Now, we use (2.1) and (2.2) to complete the proof. Let $\delta_n(\mathbf{x}) = \varphi_{\mathbf{e}_1}(\mathbf{x}) + \frac{1}{\sqrt{n}}$, then

$$\delta_n(\mathbf{x}) \sim \max\left\{ \varphi_{\mathbf{e}_1}(\mathbf{x}), \frac{1}{\sqrt{n}} \right\}, \tag{2.3}$$

where $a \sim b$ means that there exists a positive constant C, such that $C^{-1}a \leqslant b \leqslant Ca$.

If $\varphi_{\mathbf{e}_1}(\mathbf{x}) > \frac{1}{\sqrt{n}}$, then $\delta_n(\mathbf{x}) \sim \varphi_{\mathbf{e}_1}(\mathbf{x})$ and by (2.1) it follows that

$$\begin{aligned}
\left| \varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x}) \left(\frac{\partial}{\partial x_1} \right)^r B_{n,d}(f,\mathbf{x}) \right| &\leqslant Cn^{r/2}\varphi_{\mathbf{e}_1}^{r(\lambda-1)}(\mathbf{x})\|f\|_{C(S_d)} \\
&\leqslant Cn^{r/2}\delta_n^{r(\lambda-1)}(\mathbf{x})\|f\|_{C(S_d)}.
\end{aligned}$$

If $\varphi_{\mathbf{e}_1}(\mathbf{x}) \leqslant \frac{1}{\sqrt{n}}$, then $\delta_n(\mathbf{x}) \sim \frac{1}{\sqrt{n}}$ and by (2.2) it follows that

$$\begin{aligned}
&\left|\varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x}) \left(\frac{\partial}{\partial x_1}\right)^r B_{n,d}(f,\mathbf{x})\right| \\
&\quad \leqslant Cn^r \varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x}) \|f\|_{C(S_d)} \\
&\quad \leqslant Cn^{r/2} n^{r(1-\lambda)/2} \|f\|_{C(S_d)} \leqslant Cn^{r/2}\delta^{r(1-\lambda)}\|f\|_{C(S_d)}.
\end{aligned}$$

Therefore, combining the two cases and using (2.3) we have

$$\begin{aligned}
\left|\varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x}) \left(\frac{\partial}{\partial x_1}\right)^r B_{n,d}(f,\mathbf{x})\right| &\leqslant Cn^{r/2}\delta^{r(1-\lambda)}\|f\|_{C(S_d)} \\
&\leqslant Cn^{r/2}\left(\max\left(n^{-1/2}, \varphi_{\mathbf{e}_1}(\mathbf{x})\right)\right)^{r(\lambda-1)} \|f\|_{C(S_d)}.
\end{aligned}$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let* $r \in \mathbb{N}$, $0 < t < \frac{1}{8r}$, $x \pm \frac{rt}{2} \in (0,1)$ *and* $0 \leqslant \beta \leqslant r$. *Then there holds*

$$\begin{aligned}
I &= \int_{-t/2}^{t/2}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\left(\varphi\left(x+\sum_{i=1}^r u_i\right)\right)^{-\beta} du_1\,du_2\cdots du_r \\
&\leqslant Ct^r\left(\max_{0\leqslant k\leqslant r}(\varphi(x+t(r/2-k)))\right)^{-\beta}.
\end{aligned}$$

Proof. The case $\beta = 0$ is obvious, and the case $\beta = r$ has been proved by Zhou [12]. If $0 < \beta < r$, then from the Hölder inequality it follows that

$$\begin{aligned}
I &\leqslant \left(\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\varphi^{-r}\left(x+\sum_{i=1}^r u_i\right) du_1\,du_2\cdots du_r\right)^{\beta/r} \\
&\quad \times\left(\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2} du_1\,du_2\cdots du_r\right)^{1-\beta/r} \\
&\leqslant Ct^r\left(\max_{0\leqslant k\leqslant r}(\varphi(x+t(r/2-k)))\right)^{-\beta}.
\end{aligned}$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let* $f \in D_r(S)$, $r \in \mathbb{N}$, $0 \leqslant \lambda \leqslant 1$, *and* $\zeta \in V_S$. *Then we have*

$$\left|\varphi_\zeta^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial\zeta}\right)^r B_{n,d}(f,\mathbf{x})\right| \leqslant C\left\|\varphi_\zeta^{r\lambda}\left(\frac{\partial}{\partial\zeta}\right)^r f\right\|_{C(S_d)}.$$

Proof. Similar to the discussion in the proof of Lemma 2.1, we only need to prove the case $\zeta = \mathbf{e}_2$, i.e.,

$$\left|\varphi_{\mathbf{e}_2}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial x_2}\right)^r B_{n,d}(f,\mathbf{x})\right| \leqslant C\left\|\varphi_{\mathbf{e}_2}^{r\lambda}\left(\frac{\partial}{\partial x_2}\right)^r f\right\|_{C(S_d)}. \tag{2.4}$$

If $\lambda = 0$, then (2.4) follows from the fact

$$\left(\frac{\partial}{\partial x_2}\right)^r B_{n,d}(f,\mathbf{x}) = \frac{n!}{(n-r)!}\sum_{|\mathbf{k}|\leqslant n-r} P_{n-r,\mathbf{k}}(\mathbf{x})\Delta^r_{\frac{1}{n}\mathbf{e}_2} f\left(\frac{\mathbf{k}}{n}\right)$$
$$= \frac{n!}{(n-r)!}\sum_{|\mathbf{k}|\leqslant n-r} P_{n-r,\mathbf{k}}(\mathbf{x})\int_0^{1/n}\int_0^{1/n}\cdots\int_0^{1/n}\left(\frac{\partial}{\partial x_2}\right)^r$$
$$\times f\left(\frac{\mathbf{k}}{n} + \mathbf{e}_2\sum_{i=1}^r u_i\right) du_1\, du_2 \cdots du_r.$$

If $0 < \lambda \leqslant 1$, we then use the induction on the dimension number d to prove (2.4). When $d = 1$, we can write

$$\varphi^{r\lambda}(x)B_{n,1}^{(r)}(f,\mathbf{x}) = \frac{n!}{(n-r)!}\varphi^{r\lambda}(x)\sum_{k=0}^{n-r} P_{n-r,k}(x)\Delta^r_{1/n} f\left(\frac{k}{n}\right)$$
$$= \frac{n!}{(n-r)!}\varphi^{r\lambda}(x)P_{n-r,0}(x)\Delta^r_{1/n} f(0)$$
$$+\frac{n!}{(n-r)!}\varphi^{r\lambda}(x)\sum_{k=1}^{n-r-1} P_{n-r,k}(x)\Delta^r_{1/n} f\left(\frac{k}{n}\right)$$
$$+\frac{n!}{(n-r)!}\varphi^{r\lambda}(x)P_{n-r,n-r}(x)\Delta^r_{1/n} f\left(\frac{n-r}{n}\right)$$
$$:= Q_1 + Q_2 + Q_3.$$

Now, we estimate Q_1, Q_2 and Q_3, respectively. Recalling the inequalities (see p. 155 of [10])

$$\left|\Delta^r_{1/n} f\left(\frac{k}{n}\right)\right| \leqslant Cn^{-r+1}\int_0^{r/n}\left|f^{(r)}\left(\frac{k}{n}+u\right)\right| du, \quad 0 < k < n-r,$$

$$\left|\Delta^r_{1/n} f(0)\right| \leqslant Cn^{-r/2+1}\int_0^{r/n}\left|u^{r/2} f^{(r)}(u)\right| du$$

and

$$\left|\Delta^r_{1/n} f\left(\frac{n-r}{n}\right)\right| \leqslant Cn^{-r/2+1}\int_{1-r/n}^{1}\left|(1-u)^{r/2} f^{(r)}(u)\right| du,$$

we have

$$\begin{aligned}|Q_1| &\leqslant C \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]} n^{r/2+1}\varphi^{r\lambda}(x)P_{n-r,0}(x)\int_0^{r/n} \frac{u^{r(1-\lambda)/2}}{(1-u)^{r\lambda/2}}\,du \\ &\leqslant C \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]} n^{r\lambda/2}(1-x)^{n-r+r\lambda/2}x^{r\lambda/2} \\ &\leqslant C \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]}.\end{aligned}$$

Similarly,

$$|Q_3| \leqslant C \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]}.$$

For Q_2, we obtain

$$\begin{aligned}|Q_2| &\leqslant Cn \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]} \varphi^{r\lambda}(x)\sum_{k=1}^{n-r-1} P_{n-r,k}(x)\int_0^{r/n} \frac{du}{\varphi^{r\lambda}\left(\frac{k}{n}+u\right)} \\ &\leqslant Cn \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]} \varphi^{r\lambda}(x)\sum_{k=1}^{n-r-1} P_{n-r,k}(x)\int_0^{r/n} \frac{du}{\left(\frac{k}{n}\left(1-\frac{k+r}{n}\right)\right)^{r\lambda/2}} \\ &\leqslant C \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]} \left(\sum_{k=1}^{n-r-1} P_{n-r,k}(x)\varphi^{2r}(x)\left(\frac{n}{k}\right)^r\left(\frac{n}{n-r-k}\right)^r\right)^{\lambda/2} \\ &= C \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]} \left(\sum_{k=1}^{n-r-1} P_{n+r,k+r}(x)Q(n,k,r)\right)^{\lambda/2},\end{aligned}$$

where

$$Q(n,k,r) := \frac{(n-r)!}{(n+r)!}\,\frac{(k+r)!}{k!}\,\frac{(n-k)!}{(n-r-k)!}\,\frac{n^r}{k^r}\,\frac{n^r}{(n-r-k)^r} \leqslant C.$$

So,

$$Q_2 \leqslant C \left\|\varphi^{r\lambda} f^{(r)}\right\|_{C[0,1]}.$$

Hence, (2.4) is valid for $d=1$.

Next, suppose that (2.4) is valid for $d=m$, $m\geqslant 1$; we prove (2.2) is also true for $d=m+1$. To observe this, we use a decomposition technique for $B_{n,d}(f,\mathbf{x})$ used in [1]. Let

$$H_{k_1}(\mathbf{u}) := f\left(\frac{k_1}{n}, \left(1-\frac{k_1}{n}\right)\mathbf{u}\right), \quad \mathbf{u} := (u_1,u_2,\ldots,u_{d-1}) \in S_{d-1}$$

and

$$\mathbf{y} := (y_1,y_2,\ldots,y_{d-1}) := \begin{cases} \frac{\mathbf{x}^*}{1-x_1}, & 0\leqslant x_1 < 1; \\ \mathbf{0}, & x_1 = 1, \end{cases}$$

where $\mathbf{x}^* := (x_2, x_3, \ldots, x_d)$, $(x_1, \mathbf{x}^*) \in S_d$. We then have

$$B_{n,d}(f, \mathbf{x}) = \sum_{k_1=0}^{n} P_{n,k_1}(x_1) B_{n-k_1,d-1}(H_{k_1}(\cdot), \mathbf{y}),$$

and obtain

$$\varphi_{\mathbf{e}_2}^{r\lambda}(\mathbf{x}) \left(\frac{\partial}{\partial x_2}\right)^r B_{n,d}(f, \mathbf{x}) = \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(1-x_1)^{r(\lambda-1)} \times \varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{y}) \left(\frac{\partial}{\partial y_1}\right)^r B_{n-k_1,d-1}(H_{k_1}, \mathbf{y}). \tag{2.5}$$

Since the summation in (2.5) may be taken on $0 \leqslant k_1 \leqslant n-r$, we have

$$\begin{aligned}
&\left|\varphi_{\mathbf{e}_2}^{r\lambda}(\mathbf{x}) \left(\frac{\partial}{\partial x_2}\right)^r B_{n,d}(f, \mathbf{x})\right| \\
&= \left|\sum_{k_1=0}^{n-r} P_{n,k_1}(x_1)(1-x_1)^{r(\lambda-1)} \varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{y}) \left(\frac{\partial}{\partial y_1}\right)^r B_{n-k_1,d-1}(H_{k_1}, \mathbf{y})\right| \\
&\leqslant \sum_{k_1=0}^{n-r} P_{n,k_1}(x_1)(1-x_1)^{r(\lambda-1)} \left\|\varphi_{\mathbf{e}_1}^{r\lambda}(\cdot) \left(\frac{\partial}{\partial \mathbf{e}_1}\right)^r H_{k_1}(\cdot)\right\|_{C(S_{d-1})}.
\end{aligned}$$

Recalling that

$$\begin{aligned}
&\varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{u}) \left(\frac{\partial}{\partial \mathbf{e}_1}\right)^r H_{k_1}(\mathbf{u}) \\
&= (u_1(1-|\mathbf{u}|))^{r\lambda/2} \left(1-\frac{k_1}{n}\right)^r \left(\frac{\partial}{\partial \mathbf{e}_2}\right)^r f\left(\frac{k_1}{n}, \left(1-\frac{k_1}{n}\right)\mathbf{u}\right) \\
&= \left(1-\frac{k_1}{n}\right)^{r(1-\lambda)} \varphi_{\mathbf{e}_2}^{r\lambda}\left(\frac{k_1}{n}, \left(1-\frac{k_1}{n}\right)\mathbf{u}\right) \left(\frac{\partial}{\partial \mathbf{e}_2}\right)^r f\left(\frac{k_1}{n}, \left(1-\frac{k_1}{n}\right)\mathbf{u}\right),
\end{aligned}$$

we find

$$\begin{aligned}
&\left|\varphi_{\mathbf{e}_2}^{r\lambda}(\mathbf{x}) \left(\frac{\partial}{\partial x_2}\right)^r B_{n,d}(f, \mathbf{x})\right| \\
&\leqslant C \left\|\varphi_{\mathbf{e}_2}^{r\lambda} \left(\frac{\partial}{\partial x_2}\right)^r f\right\|_{C(S_d)} (1-x_1)^{r(\lambda-1)} \sum_{k_1=0}^{n-r} P_{n,k_1}(x_1) \left(1-\frac{k_1}{n}\right)^{r(1-\lambda)} \\
&\leqslant C \left\|\varphi_{\mathbf{e}_2}^{r\lambda} \left(\frac{\partial}{\partial x_2}\right)^r f\right\|_{C(S_d)} (1-x_1)^{r(\lambda-1)} \left(\sum_{k_1=0}^{n-r} P_{n,k_1}(x_1) \left(1-\frac{k_1}{n}\right)^r\right)^{1-\lambda}
\end{aligned}$$

$$= C \left\| \varphi_{\mathbf{e}_2}^{r\lambda} \left(\frac{\partial}{\partial x_2} \right)^r f \right\|_{C(S_d)} \left(\sum_{k_1=0}^{n-r} \frac{n!}{(n-r)!} \frac{(n-r-k)!}{(n-k)!} \times \frac{(n-k)^r}{n^r} P_{n-r,k_1}(x_1) \right)^{1-\lambda}$$

$$= C \left\| \varphi_{\mathbf{e}_2}^{r\lambda} \left(\frac{\partial}{\partial x_2} \right)^r f \right\|_{C(S_d)} \left(C \sum_{k_1=0}^{n-r} P_{n-r,k_1}(x_1) \right)^{1-\lambda}$$

$$= C \left\| \varphi_{\mathbf{e}_2}^{r\lambda} \left(\frac{\partial}{\partial x_2} \right)^r f \right\|_{C(S_d)} .$$

The proof of Lemma 2.3 is complete. □

Now, we define the linear combination operators of $B_{n,1}(f, x)$ as (see also section 9.2 of [10])

$$B_{n,1}(f, r, x) := \sum_{i=0}^{r-1} a_i(n) B_{n_i,1}(f, x), \quad r \in \mathbb{N},$$

where $a_i(n)$ satisfy

(a) $n = n_0 < n_1 < \cdots < n_{r-1} \leqslant Cn$;
(b) $\sum_{i=0}^{r-1} a_i(n) = 1$;
(c) $\sum_{i=0}^{r-1} |a_i(n)| \leqslant C$;
(d) $\sum_{i=0}^{r-1} a_i(n) n_i^{-k} = 0, \quad k = 1, 2, \ldots, r-1$.

The following estimate for $B_{n,1}(f, r, x)$ is similar to the corresponding ones of Ditzian [5] and Zhou [12].

Lemma 2.4. *Let $f \in C[0, 1]$, $r \in \mathbb{N}$, and $0 \leqslant \lambda \leqslant 1$. Then we have*

$$\left| B_{n,1}(f, r, x) - f(x) \right| \leqslant C \omega_r \left(f, n^{-1/2} A_n^{1-\lambda}(x) \right)_{\varphi^\lambda},$$

where $\omega_r(f, t)_{\varphi^\lambda} = \omega_r^{\mathbf{e}_1}(f, t)_{\varphi_{\mathbf{e}_1}^\lambda}$ is Ditzian–Totik's modulus of univariate function $f \in C[0, 1]$ (see [10]), and $A_n(x) := \varphi(x) + \frac{1}{\sqrt{n}}$.

Proof. For $f \in C[0, 1]$ and $0 \leqslant \lambda \leqslant 1$, we denote $K_{\varphi^\lambda}^r(f, t) := K_{\varphi_\zeta^\lambda}^r(f, t)$ and define another K-functional as

$$\overline{K}_{\varphi^\lambda}^r(f, t) := \inf_{g \in D_r[0,1]} \left\{ \|f - g\|_{C[0,1]} + t^r \left\| \varphi^{r\lambda} g^{(r)} \right\|_{C[0,1]} + t^{r/(1-\lambda/2)} \left\| g^{(r)} \right\|_{C[0,1]} \right\}.$$

It was proved in Chapters 2 and 3 of [10] that

$$\omega_r(f,t)_{\varphi^\lambda} \sim K^r_{\varphi^\lambda}(f,t) \sim \overline{K}^r_{\varphi^\lambda}(f,t), \tag{2.6}$$

A method similar to Lemma 5.3 of [7] (see also p. 141 of [10]) gives that for u between t and x there hold

$$\frac{|t-u|^{r-1}}{\varphi^{\lambda r}(u)} \leqslant \frac{|t-x|^{r-1}}{\varphi^{\lambda r}(x)}, \qquad \frac{|t-u|^{r-1}}{A_n^{\lambda r}(u)} \leqslant \frac{|t-x|^{r-1}}{A_n^{\lambda r}(x)}.$$

Then, for any $g \in D_r[0,1]$, using Taylor's formula

$$g(t)-g(x) = \sum_{k=0}^{r-1} \frac{g^{(k)}(x)}{k!}(t-x)^k + \frac{1}{(r-1)!}\int_x^t (t-u)^{r-1} g^{(r)}(u)\,du$$

and the fact that $B_{n,1}((\cdot - x)^k, r, x) = 0$ for $k = 1, 2, \ldots, r-1$ (see Section 9.2 of [10]), we see

$$\begin{aligned}
&\left|B_{n,1}(g,r,x)-g(x)\right| \\
&\quad\leqslant \frac{1}{(r-1)!} B_{n,1}\left(\left|\int_x^t (t-u)^{r-1} g^{(r)}(u)\,du\right|, r, x\right) \\
&\quad\leqslant \left\|A_n^{r\lambda} g^{(r)}\right\|_{C[0,1]} A_n^{-r\lambda}(x) \sum_{i=0}^{r-1} |a_i(n)| B_{n_i,1}(|t-x|^r, x) \\
&\quad\leqslant \left\|A_n^{r\lambda} g^{(r)}\right\|_{C[0,1]} A_n^{-\lambda r}(x) \sum_{i=0}^{r-1} |a_i(n)| \left(B_{n_i,1}(|t-x|^{2r}, x)\right)^{1/2} \\
&\quad\leqslant C \left\|A_n^{r\lambda} g^{(r)}\right\|_{C[0,1]} A_n^{-\lambda r}(x) \sum_{i=0}^{r-1} |a_i(n)| \left(\frac{\varphi^2(x)}{n_i} + \frac{1}{n_i^2}\right)^{r/2} \\
&\quad\leqslant C n^{-r/2} \left\|A_n^{r\lambda} g^{(r)}\right\|_{C[0,1]} A_n^{r(1-\lambda)}(x).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\left|B_{n,1}(g,r,x)-g(x)\right| &\leqslant \left\|\varphi^{r\lambda} g^{(r)}\right\|_{C[0,1]} \varphi^{-r\lambda}(x) \sum_{i=0}^{r-1} |a_i(n)| B_{n_i,1}\left(|t-x|^r, x\right) \\
&\leqslant C n^{-r/2} \left\|\varphi^{r\lambda} g^{(r)}\right\|_{C[0,1]} A_n^r(x) \varphi^{-r\lambda}(x).
\end{aligned}$$

Thus, for $\varphi(x) \geqslant 1/\sqrt{n}$, $f \in C[0,1]$, then $A_n(x) \sim \varphi(x)$, and

$$\begin{aligned}
\left|B_{n,1}(f,r,x)-f(x)\right| &\leqslant C\left\{\|f-g\|_{C[0,1]} + n^{-r/2} A_n^r(x)\varphi^{-r\lambda}(x)\left\|\varphi^{\lambda r} g^{(r)}\right\|_{C[0,1]}\right\} \\
&\leqslant C\left\{\|f-g\|_{C[0,1]} + n^{-r/2} A_n^{r(1-\lambda)}(x)\left\|\varphi^{\lambda r} g^{(r)}\right\|_{C[0,1]}\right\},
\end{aligned}$$

which implies from (2.6) that

$$\left|B_{n,1}(f,r,x)-f(x)\right| \leqslant CK^r_{\varphi^\lambda}\left(f, n^{-1/2}A_n^{1-\lambda}(x)\right) \leqslant C\omega_r\left(f, n^{-1/2}A_n^{1-\lambda}(x)\right)_{\varphi^\lambda}.$$

For $\varphi(x)\leqslant 1/\sqrt{n}$, then $A_n(x)\sim 1/\sqrt{n}$, and

$$\begin{aligned}
&\left|B_{n,1}(f,r,x)-f(x)\right| \\
&\leqslant C\left\{\|f-g\|_{C[0,1]}+n^{-r/2}A_n^{r(1-\lambda)}(x)\left\|A_n^{\lambda r}g^{(r)}\right\|_{C[0,1]}\right\} \\
&\leqslant C\left\{\|f-g\|_{C[0,1]}+n^{-r/2}A_n^{r(1-\lambda)}(x)\right. \\
&\quad\times\left.\left(\left\|\varphi^{\lambda r}g^{(r)}\right\|_{C[0,1]}+n^{-(r\lambda)/2}\left\|g^{(r)}\right\|_{C[0,1]}\right)\right\} \\
&\leqslant C\left\{\|f-g\|_{C[0,1]}+n^{-r/2}A_n^{r(1-\lambda)}(x)\left\|\varphi^{\lambda r}g^{(r)}\right\|_{C[0,1]}\right. \\
&\quad+\left.\left(n^{-1/2}A_n^{1-\lambda}(x)\right)^{r/(1-\lambda/2)}\left\|g^{(r)}\right\|_{C[0,1]}\right\},
\end{aligned}$$

which implies from (2.6) that

$$\left|B_{n,1}(f,r,x)-f(x)\right| \leqslant C\omega_r\left(f, n^{-1/2}A_n^{1-\lambda}(x)\right)_{\varphi^\lambda}.$$

Combining the above two cases we complete the proof of Lemma 2.4. □

Remark 2.5. In the case $\lambda=0$, Lemma 2.4 is Theorem 2.1 of [12]. In the sequel, we only use the conclusion of case $\lambda=0$, but we expect the lemma to be conducive to answer the inverse of Theorem 1.2.

Lemma 2.6. *For $f\in C(S)$, $\zeta\in V_S$, $r\in\mathbb{N}$, and $0\leqslant\lambda\leqslant 1$, we have*

$$\omega_r^\zeta(f,t)_{\varphi_\zeta^\lambda}\sim K_r^\zeta(f,t)_{\varphi_\zeta^\lambda}. \tag{2.7}$$

Proof. The case $d=1$ was proved by Theorem 2.1.1 of [10]. For $d>1$, we can show (2.7) by a decomposition technique used in Theorem 1 of [1]. We omit the details. □

3. Proof of main result

We first prove Theorem 1.2. For $\zeta = \mathbf{e}_1$ and any $g \in D_r(S)$, from Lemmas 2.1, 2.3 and (2.3) it follows that

$$\begin{aligned}
&\left|\varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial x_1}\right)^r B_{n,d}(f,\mathbf{x})\right| \\
&\quad\leqslant \left|\varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial x_1}\right)^r B_{n,d}(f-g,\mathbf{x})\right| + \left|\varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial x_1}\right)^r B_{n,d}(g,x)\right| \\
&\quad\leqslant Cn^{r/2}\left(\min(n^{1/2},\varphi_{\mathbf{e}_1}^{-1}(\mathbf{x}))\right)^{r(1-\lambda)}\|f-g\|_{C(S_d)} + C\left\|\varphi_{\mathbf{e}_1}^{r\lambda}\left(\frac{\partial}{\partial x_1}\right)^r g\right\|_{C(S_d)}.
\end{aligned}$$

Thus, by the definition of K-functional and Lemma 2.6, we derive that

$$\begin{aligned}
&\left|\varphi_{\mathbf{e}_1}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial x_1}\right)^r B_{n,d}(f,\mathbf{x})\right| \\
&\quad\leqslant C\left(\min\left(n^{2-\lambda},\frac{n}{\varphi_{\mathbf{e}_1}^{2(1-\lambda)}(\mathbf{x})}\right)\right)^{r/2} K_r^{\mathbf{e}_1}\left(f,\left(\min\left\{n^{1-\lambda/2},\frac{\sqrt{n}}{\varphi_{\mathbf{e}_1}^{1-\lambda}(\mathbf{x})}\right\}\right)^{-1}\right)_{\varphi_{\mathbf{e}_1}^{\lambda}} \\
&\quad\leqslant C\left(\min\left(n^{2-\lambda},\frac{n}{\varphi_{\mathbf{e}_1}^{2(1-\lambda)}(\mathbf{x})}\right)\right)^{r/2} \omega_r^{\mathbf{e}_1}\left(f,\left(\min\left\{n^{1-\lambda/2},\frac{\sqrt{n}}{\varphi_{\mathbf{e}_1}^{1-\lambda}(\mathbf{x})}\right\}\right)^{-1}\right)_{\varphi_{\mathbf{e}_1}^{\lambda}} \\
&\quad\leqslant C\left\{\min\left(n^{2-\lambda},\frac{n}{\varphi_{\mathbf{e}_1}^{2(1-\lambda)}(\mathbf{x})}\right)\right\}^{(r-\alpha)/2}.
\end{aligned}$$

Similarly, the cases for $\zeta = \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_d$ can be proved. If $\zeta = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}$, $1 \leqslant i < j \leqslant d$, then it is not difficult to obtain for $\eta = \mathbf{e}_i$ and $\mathbf{u} = T\mathbf{x}$ that

$$\begin{aligned}
\omega_r^{\zeta}(f,h)_{\varphi_\zeta^\lambda} &= \sup_{0<t\leqslant h}\left\|\Delta^r_{t\varphi_\zeta^\lambda(\mathbf{x})\zeta} f(\mathbf{x})\right\|_{C(S_d)} = \sup_{0<t\leqslant h}\left\|\Delta^r_{t\varphi_\eta^\lambda(\mathbf{u})\eta} f_T(\mathbf{u})\right\|_{C(S_d)} \\
&= \omega_r^{\eta}(f_T,h)_{\varphi_\eta^\lambda}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\varphi_{\zeta}^{r\lambda}(\mathbf{x})\left(\frac{\partial}{\partial \zeta}\right)^r B_{n,d}(f,\mathbf{x})\right| &= \left|\varphi_{\eta}^{r\lambda}(\mathbf{u})\left(\frac{\partial}{\partial \eta}\right)^r B_{n,d}(f_T,\mathbf{u})\right| \\
&\leqslant C\left\{\min\left(n^{2-\lambda},\ \frac{n}{\varphi_{\eta}^{2(1-\lambda)}(\mathbf{u})}\right)\right\}^{(r-\alpha)/2} \\
&= C\left\{\min\left(n^{2-\lambda},\ \frac{n}{\varphi_{\zeta}^{2(1-\lambda)}(\mathbf{x})}\right)\right\}^{(r-\alpha)/2}.
\end{aligned}$$

Next, we prove Theorem 1.3. From Theorem 1.2 (the case $\lambda = 0$) the sufficiency is obvious. To show the necessary, we define the linear combination operators of $B_{n,d} f$ as

$$B_{n,d}(f, r, \mathbf{x}) := \sum_{i=0}^{r-1} a_i(n) B_{n_i,d}(f, \mathbf{x}),$$

where the coefficients $a_i(n)$ satisfy the conditions (a)–(d) given in Section 2. For $\mathbf{x} := (x_1, \mathbf{x}^*) \in S_d$, let $\mathbf{x}^* = (x_2, x_3, \ldots, x_d)$ be fixed, and

$$F(t) := f(t, \mathbf{x}^*), \quad t \in [0, 1], \quad (t, \mathbf{x}^*) \in S_d,$$

then

$$\begin{aligned} B_{n,d}(F, \mathbf{x}) &= \sum_{|\mathbf{k}| \leqslant n} P_{n,\mathbf{k}}(\mathbf{x}) f\left(\frac{k_1}{n}, \mathbf{x}^*\right) \\ &= \sum_{k_1=0}^{n} P_{n,k_1}(x_1) f\left(\frac{k_1}{n}, \mathbf{x}^*\right) = B_{n,1}(F(\cdot), x_1). \end{aligned}$$

Therefore, from Lemma 2.4 it follows that

$$\begin{aligned} |B_{n,d}(F, r, \mathbf{x}) - f(\mathbf{x})| &= |B_{n,1}(F, r, x_1) - F(x_1)| \\ &\leqslant C\omega_r\left(F, n^{-1/2} A_n^{1-\lambda}(x_1)\right). \end{aligned} \tag{3.1}$$

Let $h \in (0, 1/8r)$, $0 < t \leqslant h$, $x_1, x_1 + rt \in (0, 1)$, then

$$\begin{aligned} \left|\Delta_{t\mathbf{e}_1}^r f(\mathbf{x})\right| &\leqslant \left|\Delta_{t\mathbf{e}_1}^r \left(f(\mathbf{x}) - B_{n,d}(F, r, \mathbf{x})\right)\right| + \left|\Delta_{t\mathbf{e}_1}^r \left(B_{n,d}(F, r, \mathbf{x})\right)\right| \\ &:= I_1 + I_2. \end{aligned}$$

From (3.1), we obtain

$$\begin{aligned} I_1 \leqslant C \left|\Delta_{t\mathbf{e}_1}^r \omega_r\left(F, n^{-1/2} A_n(x_1)\right)\right| &\leqslant C\omega_r^{\mathbf{e}_1}(F, d(n, x_1, t)) \\ &\leqslant C\omega_r^{\mathbf{e}_1}(f, d(n, x_1, t)), \end{aligned}$$

where

$$d(n, x_1, t) := \max\left\{\frac{1}{n}, \max_{0 \leqslant k \leqslant r} \frac{\varphi(x_1 + (r-k)t)}{\sqrt{n}}\right\}.$$

Using the definition of linear combination operators we have

$$\begin{aligned} I_2 &\leqslant \int_0^t \cdots \int_0^t \left|\left(\frac{\partial}{\partial \mathbf{e}_1}\right)^r B_{n,d}\left(f(\cdot, \mathbf{x}^*), r, \left(x_1 + \sum_{i=1}^r y_i, \mathbf{x}^*\right)\right)\right| dy_1 \cdots dy_r \\ &= \int_0^t \cdots \int_0^t \left|\sum_{i=0}^{r-1} a_i(n) B_{n_i,1}^{(r)}\left(F, x_1 + \sum_{i=1}^r y_i\right)\right| dy_1 \cdots dy_r \end{aligned}$$

$$\leqslant C \int_0^t \cdots \int_0^t \sum_{i=0}^{r-1} |a_i(n)| \left(\min \left(n_i^2, \ \frac{n_i}{\varphi^2 \left(x_1 + \sum_{i=1}^r y_i \right)} \right) \right)^{(r-\alpha)/2}$$
$$\times dy_1 \cdots dy_r$$
$$\leqslant C \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \left(\min \left(n^2, \ \frac{n}{\varphi^2 \left(x_1 + rt/2 + \sum_{i=1}^r \ y_i \right)} \right) \right)^{(r-\alpha)/2}$$
$$\times dy_1 \cdots dy_r.$$

Thus, on the one hand, we have

$$I_2 \leqslant C t^r n^{r-\alpha},$$

and on the other, from Lemma 2.2 it follows that

$$I_2 \leqslant C n^{(r-\alpha)/2} \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \varphi^{-r+\alpha} \left(x_1 + \frac{rt}{2} + \sum_{i=1}^r \ y_i \right) dy_1 \cdots dy_r$$
$$\leqslant C n^{(r-\alpha)/2} t^r \left(\max_{0 \leqslant k \leqslant r} \ \varphi \left(x_1 + \frac{rt}{2} + \left(\frac{r}{2} - k \right) t \right) \right)^{-r+\alpha}$$
$$= C t^r \left(\frac{\sqrt{n}}{\max_{0 \leqslant k \leqslant r} \ \varphi \left(x_1 + (r-k)t \right)} \right)^{r-\alpha}.$$

Thus,

$$I_2 \leqslant C t^r \ (d(n, x_1, t))^{\alpha - r} \ ,$$

and

$$\left| \Delta_{t\mathbf{e}_1}^r f(\mathbf{x}) \right| \leqslant C \omega_r^{\mathbf{e}_1} \ (f, d(n, x_1, t)) + C t^r \ (d(n, x_1, t))^{\alpha - r} \ .$$

Now, we use a recursion method used in [12] to imply

$$\omega_r^{\mathbf{e}_1}(f, h) = \mathcal{O} \left(h^\alpha \right). \tag{3.2}$$

For any $\delta \in \left(0, \frac{1}{8r}\right)$, we can choose an n, such that

$$d(n, x_1, t) \leqslant \delta < d(n-1, x_1, t) \leqslant 2d(n, x_1, t).$$

Consequently,

$$\left| \Delta_{t\mathbf{e}_1}^r f(\mathbf{x}) \right| \leqslant C_1 \omega_r^{\mathbf{e}_1}(f, \delta) + C_2 t^r \delta^{\alpha - r}$$

with the constants C_1 and C_2 independent of $\mathbf{x}, t, h$ and δ, and $C_1, C_2 > 1$. Using the condition $\omega_r^{\mathbf{e}_1}(f, h) = \mathcal{O}\left(h^\beta\right)$ with some $\beta > 0$ and letting

$$A = (2C_1 + 1)^{\alpha^{-1} + \beta^{-1}} \ , \quad \delta = \frac{h}{A},$$

we obtain for any $k \in \mathbb{N}$

$$\begin{aligned}
\omega_r^{\mathbf{e}_1}(f,h) &\leqslant C_1\omega_r^{\mathbf{e}_1}(f,h/A) + C_2A^{r-\alpha}h^\alpha \\
&\leqslant C_1^2\omega_r^{\mathbf{e}_1}\left(f,h/A^2\right) + C_1C_2A^{r-2\alpha}h^\alpha + C_2A^{r-\alpha}h^\alpha \\
&\leqslant \cdots \\
&\leqslant C_1^k\omega_r^{\mathbf{e}_1}\left(f,h/A^k\right) + C_2h^\alpha A^{r-\alpha}\sum_{l=0}^{k-1}\left(C_1A^{-\alpha}\right)^l \\
&\leqslant CC_1^kh^\beta A^{-\beta k} + C_2h^\alpha A^{r-\alpha}\frac{\left(1-\left(C_1A^{-\alpha}\right)^k\right)}{1-C_1A^{-\alpha}} \\
&\leqslant Ch^\beta\left(\frac{C_1}{(2C_1+1)^{1+\beta/\alpha}}\right)^k + Ch^\alpha A^{r-\alpha}\left(1-\left(\frac{C_1}{(2C_1+1)^{1+\alpha/\beta}}\right)^k\right).
\end{aligned}$$

Thus, letting $k \to \infty$, we derive (3.2)

Also, for $\zeta = \mathbf{e}_i$, $i = 2, 3, \ldots, d$, we can obtain

$$\omega_r^{\mathbf{e}_i}(f,h) = \mathcal{O}\left(h^\alpha\right).$$

For $\zeta = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}$, $1 \leqslant i < j \leqslant d$, set $\eta = \mathbf{e}_i$, $\mathbf{u} = T\mathbf{x}$, we have

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial\eta}\right)^r B_{n,d}(f_T,\mathbf{u})\right| = \left|\left(\frac{\partial}{\partial\zeta}\right)^r B_{n,d}(f,\mathbf{x})\right|
&\leqslant C\left\{\min\left(n^2, n/\varphi_\zeta^2(\mathbf{x})\right)\right\}^{(r-\alpha)/2} \\
&= C\left\{\min\left(n^2, n/\varphi_\eta^2(\mathbf{u})\right)\right\}^{(r-\alpha)/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\omega_r^\zeta(f,h) &= \sup_{0<t\leqslant h}\left\|\Delta_{t\zeta}^r f(\mathbf{x})\right\|_{C(S_d)} = \sup_{0<t\leqslant h}\left\|\Delta_{t\eta}^r f_T(\mathbf{u})\right\|_{C(S_d)} = \omega_r^\eta(f_T,h) \\
&= \mathcal{O}\left(h^\alpha\right).
\end{aligned}$$

The proof of Theorem 1.3 is complete.

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